Tutorial: <https://uofglasgow.zoom.us/j/95534070147?pwd=M2kzSHF5V2MwTmFCV2J0QlEzckFXUT09>

# Week 1

## 2B: Linear Combinations, SoLE, Matrix

A vector is a **linear combination** of vectors if there are scalars to make

A **linear equation** is an equation that can be written in the form

* The solution is a vector with the values for x’s

A System of Linear Equations (**SoLE**):

* **Consistent** if
  + There is a unique solution
  + There are infinitely many solutions
* **Inconsistent** if
  + There is no solution
* Augmented matrix – with all coefficients and also the free variables (constants)
* **Homogeneous** if all constants are 0

A matrix is in **row echelon form** if and only if

1. any all-zero rows are at the bottom
2. in each non-zero row, the first non-zero entry (leading entry) is to the left of any leading entries below it

***Reduced* row echelon form**:

1. in row echelon form
2. the leading entry in all non-zero rows is 1
3. each column containing a leading 1 has zeros everywhere else

### Spanning sets

* A **span** is called the set of all possible linear combinations using a set of certain given vectors
* A **spanning set** for if and only if span(S)
  + To prove that a set is spanning, prove that you can find coefficients to make any arbitrary vector

### Linear Independence

* A set of vectors is *linearly independent* if and only if the only solution to the equation
* is
* The set is linearly **de**pendent if and only if at least one of them can be expressed as a linear combination of others
* Any set of vectors containing the zero vector is linearly dependent
* Lemma:
  + Two vectors are linearly dependent if and only if they are scalar multiples of each other
* Let with vectors in be matrix. Then the vectors are linearly dependent if and only if the homogeneous linear system with augmented matrix [A|**0**] has a non-trivial solution
  + If then any set of *m* vectors in is linearly dependent

# Week 2

## 2B: Matrix Operations, Linear Independence

Matrix – a rectangular array of numbers

**Diagonal** matrix – a square matrix with all off-diagonal entries are zero (for ).   
**Identity** matrix – a diagonal matrix with all diagonal entries being 1

Matrix multiplication: If A is an matrix and B is an matrix, then is an

matrix.

**Transpose** of an matrix A is the matrix

* A square matrix is *symmetric* if

Linear combinations can also be made from matrices in the same way as vectors

* Span too
* Linear independence too

## 2B: Matrix Multiplication and Inversion

Properties of **matrix multiplication**:

* – associativity
* – left distributivity
* – right distributivity
* scalar multiples can be put anywhere
* Multiplying with the identity matrix (any side) doesn’t change anything

**Transpose** properties:

* for all integers
* If A is a square matrix, then – symmetric
* and – symmetric

**Inverse**:

* If A is an n x n matrix, the inverse of A is an n x n matrix A’ such that
* and
* If A’ exists, A is invertible
  + Then the inverse is **unique**
* The inverse is written as
* If A is invertible, then the SoLE given by has the unique solution given by
* Formula:
* where |A| - determinant of a matrix. For 2x2 matrices,
  + A is invertible if the determinant isn’t 0
* Properties:

# Week 3

## 2B: Matrices

**Elementary** matrices:

* \* - a matrix which can be obtained by performing one elementary row operation (ERO) on an identity matrix)
* Performing EROs is entirely equivalent to left multiplication by elementary matrices
* If E is an elementary matrix with some ERO performed on and the same ERO is performed on some matrix A, the result is matrix EA.
* Each elementary matrix is invertible, and its inverse is an elementary matrix of the same type
  + Multiplication: reverse is division
  + Addition: reverse is subtraction
  + Interchange: reverse is that same interchange

**Invertibility**:

* The **fundamental theorem of invertible matrices**:
  + The following statements are equivalent for an matrix A:
    - A is invertible
    - has a unique solution for every **b** in
    - has only the trivial solution
    - The reduced echelon form of A is
    - A is a product of elementary matrices
* A one-sided inverse is a two-sided inverse
  + If A and B are square matrices s.t. or , then A is invertible and
* If A is a square matrix and a sequence of EROs reduces A to I, then the same sequence reduces I to
  + Gauss-Jordan method for finding the inverse:
    - A square matrix is invertible if and only if there are EROs transforming A to I. In this case, these EROS compute as

## 2B: Vector, Matrix Subspaces

A **subspace** of is a collection S of vectors in such that

* The zero vector is in S
* If **u** and **v** are in S, then are in S
* If **u** is in S and c is a real scalar, then c**u** is in S

Ex: Lines, planes (going through the origin) are subspaces of , spaces are subspaces of themselves

A line/plane not through the origin is not a subspace since it does not contain **0**

A span of a set of vectors is a subspace

If A is an matrix:

* The row space of A (row(A)) is the subspace of spanned by the rows of A
* The column space of A (col(A)) is the subspace of spanned by the columns of A

Properties:

Theorems/Properties:

* If B is a matrix equivalent to A,
  + Proof: EROs comply with the definition of a subspace
    - Interchange -> order doesn’t matter
    - Sum -> 2nd property of subspaces
    - Multiply -> 3rd property of subspaces
* If A is an matrix and N is the **set of solutions** to the homogeneous linear system then N is a subspace of . This is called the **null space** of A (null(A)).

For any system of linear equations, exactly one of the following is true:

* There is no solution
* There is a unique solution
* There are infinitely many solutions

# Week 4

## 2B: Basis, Dimension

**Basis** for a subspace S of is a set of vectors in S that

1. spans S
2. is linearly independent

* Standard basis – standard unit vectors (<1,0> and <0,1> for R2)
* Basis theorem:
  + Any two bases for the same subspace have the same number of vectors
    - Proof (examinable):
      * Assume two bases with different number of vectors (r < s)
      * Create linear independence equation with vectors from bigger basis
      * Express the vectors as a linear combination of vectors from smaller basis
      * Factor out smaller vectors
      * Since smaller basis is linearly independent, all coefficient sums need to be 0
      * r homogeneous equations in s variables => not linearly independent => contradiction (must be greater or equal)
      * Prove for the other inequality the same way

**Dimension**:

* If S is a nontrivial subspace of , then the **number of vectors in a basis** for S is called the **dimension** of S (dim(S)). If S – {**0**}, then we define dim(S) – 0.
  + For any matrix A, **dim(row(A)) = dim(col(A))**

## 2B: Rank, Nullity, Coordinates

The rank of a matrix A (rank(A)) is the dimension of its row and column spaces.

* rank(A) = rank(AT)

The nullity of a matrix A (nullity(A)) is the dimension of its null space

If A is :

* Proof (examinable):
  + If R is the reduced row echelon form of A and rank(A)=r, then rank(A)=rank(R)
  + R has r leading 1’s
  + There are r leading variables and n-r free variables in the solution
  + The nullity(A)=n-r
  + Thus, see above

Ordered bases

* Theorem: If S is a subspace of and is an ordered basis for S, then for every vector in S, there is **exactly one way to write it as an ordered linear combination** of the basis vectors in
  + Proof (examinable):
    - Since is a basis for S, it spans it
    - For any **v** in S, there exist real scalars such that
      * Thus, it can always be expressed (not unique)
    - Suppose also satisfy same thing
    - Since is a basis, it is linearly dependent. Thus
    - Thus, uniqueness. See above

Coordinates: If S is a subspace and is an ordered basis vector, and **v** in S is a linear combination with scalars , then those scalars are called the **coordinates of v with respect to** . The column vector

is called the **coordinate vector of v with respect to** .

# Week 5

## 2B: Change of Basis, Gauss-Jordan

If and are ordered bases for , the n x n matrix whose columns are the coordinate vectors

of the vectors in with respect to the basis is denoted and called the change of basis matrix from to .

Properties:

* For all
* is the unique n x n matrix P such that
* is invertible and the inverse is [P to B]

Gauss-Jordan method

## 2B: Linear Maps (Transformations)

A linear transformation from Rn to Rm is a mapping such that for all v1, v2 in Rn and scalars c in R,

* Properties
  + Every matrix transformation is a linear transformation
  + Zero map: maps to 0 vector
  + Identity transformation: maps to itself
  + If
    - for
    - for
  + Bases
    - If and B a basis with v\_1,…,v\_n for Rn, then T is completely determined by the effect on B. More precisely, if v in Rn has
    - then
    - Let P be a basis for Rm with u\_1,…,u\_m. Then the matrix of T with respect to these bases is defined to be

Composition:

* Let and be transformations. The composition is the mapping defined by
* for
* Property:
  + If T and S are linear transformations, their composition is also linear
* Proof (examinable)
  + Show summation, scalar multiplication is moved linearly

Inverses:

* A linear transformation is invertible if there exists a linear transformation with (identity map – does nothing) and . This T’ is the inverse of T.
* If T is invertible, its inverse is unique
* Proof: see for matrices
  + Assume 2 inverses
  + Contradiction by combining
* Notation: inverse is
* Properties:
  + Let be an invertible linear transformation. Then the matrices and with respect to any basis of are also inverse:

# Week 6

## 2B: Determinants

If A is 3 x 3, we define its determinant as

In general, for 3 x 3, we write for the submatrix of A obtained by deleting row i and column j. Then

Thus, the **general formula:**

For any row i, replace “1” with “i” to expand along row i.

For any column j, replace previous j=1 with i=1 to expand along column j

The determinant of an upper/lower triangular matrix is the product of the entries on its main diagonal

* Upper triangular – all non-zero elements in top right triangle
* Lower triangular – all non-zero elements in bottom left triangle

## 2B: Determinant Properties, Eigenvalues & Eigenvectors

Properties of determinants (given square A):

* If B is A but with any two ~~adjacent~~ (i.e., not necessarily adjacent) rows swapped, det(B) = - det(A)
  + Elementary: If E results from swapping two rows of I\_n, then det(E) = -1
* If B is A but with a row multiplied by k, det(B) = kdet(A)
  + Elementary: If E results from multiplying one row by non-zero k, then det(E) = k
* If B is A but with a multiple of a row added to another row, det(B) = det(A)
  + Elementary: If E results from adding a multiple of one row of I\_n, then det(E) = 1

If B and E are n x n,

If

A (n x n) is invertible if and only if det(A) =/= 0.

* Proof (examinable)
  + det(R) =/= 0 ⬄ a leading 1 in every row
  + ⬄ R = I\_n
  + ⬄ A is invertible

More properties:

* (if A is invertible)

Let A be an matrix. A scalar is called an **eigenvalue** of A if there is a non-zero vector such that

Such a vector x is called an **eigenvector** of A corresponding to .

* The eigenvalues of a square matrix A are the solutions of the equation
  + The LHS is called the **characteristic polynomial of A**
    - For n x n matrix A, its char. poly. has degree n
    - Since a degree n polynomial has at most n distinct roots, an n x n matrix A has at most n distinct eigenvalues

Finding eigenvalues and eigenvectors:

1. Compute characteristic polynomial
2. Solve equation to find eigenvalues
3. For each eigenvalue, find null space of characteristic polynomial (called eigenspace ). The lambda-eigenvectors of A are the non-zero vectors in the eigenspace

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation (how many times it appears in the equation). The **geometric multiplicity** of an eigenvalue is the dimension of its eigenspace.

* geometric m. algebraic m.

The eigenvalues of an upper- or lower-triangular matrix A are the entries on its main diagonal

A square matrix A is invertible if and only if 0 is **not** an eigenvalue of A

Let A be an n x n matrix and let be **distinct** eigenvalues of A with corresponding eigenvectors Then is linearly independent.

# Week 7

## 2B: Similarity

A and B (n x n matrices) are **similar** if there exists an invertible n x n matrix P such that

* Basic Properties
  + If , then
  + If and , then
* Properties for similar matrices A and B
  + - Proof (examinable):
      * Definition of similarity
      * definitions of determinants
  + A is invertible if and only if B is invertible
  + (same characteristic polynomial)
    - Proof (examinable):
      * Definition of similarity
      * Replace A with char. poly.
      * Open brackets
      * Use definition of similarity
      * Use first property (same determinants)
  + (same eigenvalues)

## 2B: Diagonalisation

Def:

* An n x n matrix is diagonalisable if it is similar to a diagonal matrix (entries outside the main diagonal are all zero)
  + Rephrase: A is diagonalisable if there exists an invertible matrix P and a diagonal matrix D such that
* Theorem:
  + A is diagonalisable if and only if it has n linearly independent eigenvectors. There exists an invertible matrix P and a diagonal matrix D such that if and only if:
    - the columns of P are n linearly independent eigenvectors of A;
    - the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P (in the same order)
  + Proof:
    - One way:
    - Suppose A is diagonalisable, and P is invertible such that
    - AP=PD (1)
    - Let P be matrix with column vectors
    - Let D be matrix with diagonal entries
    - From equation 1:
      * LHS = matrix with column vectors
      * RHS = matrix with column vectors
    - Then the columns of P are eigenvectors of A with corresponding eigenvalues being the entries in D (written in the same order). The columns of P are linearly independent since P is invertible. So A has n linearly independent eigenvectors.
    - Other way:
    - Suppose A has n linearly independent eigenvectors with corresponding eigenvalues
    - Then
    - Let P be matrix with column vectors
    - [Reverse steps from first part of proof to get AP=PD]
    - Since columns of P are linearly independent, P is invertible
    - Then
    - Then A is similar to D, thus: diagonalisable
  + Property:
    - A is diagonalisable if and only if there is a basis of R^n consisting of eigenvectors for A
* Theorem:
  + An n x n matrix is diagonalisable if it has n distinct eigenvalues
    - Proof: The eigenvectors corresponding to the n-distinct eigenvalues are linearly independent, so the result follows
* Diagonalisation theorem:
  + If A is an n x n matrix with distinct eigenvalues , the following are equivalent:
    - A is diagonalisable;
    - The union of a basis for each of the eigenspaces of A contains n vectors;
    - The algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity and the sum of these multiplicities across all eigenvalues is n.
* Useful property:
  + where

# Week 8

## 2B: Inner Products & Norms, Orthogonality & Gram-Schmidt

Properties of inner products:

* Commutativity
* Distributivity
* (scalar inside and outside brackets)
* Proofs:
  + using definition and properties of real numbers

The length/norm of a vector v is the non-negative defined by

* Properties:
  + if and only if **v** = **0**
    - Proof:
      * Square LHS, rearrange, square root

The Cauchy-Schwartz Inequality:

* There is equality if and only if u and v are linearly independent
* Proof:
  + 0 case: trivial
  + For any real t we have
  + which means that there is no real root or a repeated root
  + Then
  + Then
  + Then, using the fact that both factors in RHS are positive, square root and get Inequality

Triangle Inequality:

**Orthogonality**:

The angle between two vectors is

where

A set of vectors is an orthogonal set if

* An orthogonal set of non-zero vectors is a linearly independent set
  + Proof:
    - linear equation
    - multiply (dot) with v\_i
    - c\_i = 0

A basis is orthogonal if and only if it is an orthogonal set

Theorem:

* If S is an orthogonal basis for subspace V of R^n with vectors v\_1,…,v\_k. then for any v in V there are real c\_1,…,c\_k such that
* where
* for
  + Proof:
    - linear equation
    - multiply (dot) with v\_i
    - Since orthogonal, everything is 0 except v\_i \* v\_i
    - Then … as required

**Orthonormal** set: orthogonal sectors with all vectors’ norms being 1

**Gram-Schmidt Process**:

* Let be a basis for a subspace W of . Define
* The vectors are an orthogonal basis for W. If then are an orthonormal basis for W.

## 2B: Orthogonal Matrices

An **orthogonal matrix** is an n x n matrix whose columns form an **orthonormal set**

A square matrix Q is orthogonal if and only if

* Proof:
  + Aim: prove
  + Let be the i-th column of Q and i-th row of
  + The (I,j) entry of is the dot product of the i-th row of and the j-th column of Q:
  + Since the columns of Q form an orthonormal set if and only if
  + Thus, this is the identity matrix

Equivalent statements:

1. Q is orthogonal
   * From b)
   * If e\_i is the i-th standard basis vector for R^n, then
   * Consequently,
   * Thus, the columns of Q form an orthonormal set, so Q is an orthogonal matrix
   * From c):
   * From a):

* (every orthogonal matrix is an isometry, i.e. the matrix transformation preserves length)

Properties of orthogonal matrices:

* is orthogonal
  + Thus, as required
* If is an eigenvalue of Q, then
  + Since we have …, as required
* If Q\_1 and Q\_2 are orthogonal n x n matrices, so is

# Week 9

## 2B: Orthogonal Diagonalisation

An n x n matrix A is **orthogonally diagonalisable** if there exists an orthogonal matrix Q and a diagonal matrix D such that

If A is orthogonally diagonalisable, then A is symmetric.

* Proof:
  + Definition of orthogonally diagonalisable
  + Then
  + Thus, symmetric

Remarks:

* Complex conjugate of is
* If z is real,
* If is an n x n matrix, then the complex conjugate is

If A is a real symmetric matrix then

1. The eigenvalues of A are all real
2. Eigenvectors from different eigenspaces are orthogonal

* Proof:
  + Suppose – eigenvalue, – eigenvector. Then
  + , but since A is real,
  + Taking transpose (and because it’s symmetric):
  + Multiply on the right by v

## 2B: Spectral Theorem, Quadratic Form

**Spectral Theorem**:

* Let A be an n x n matrix. Then A is symmetric if and only if A is orthogonally diagonalisable
  + “Orthogonally diagonalisable symmetric” proved in [2B: Orthogonal Diagonalisation](#_2B:_Orthogonal_Diagonalisation)
  + Other way – proof by induction:
    - Definition of symmetry:
    - base: n = 1
      * Nothing to prove because 1 x 1 matrices are diagonal
    - Inductive hypothesis: Assume true for n=k, i.e. assume every k x k matric that is real and symmetric is also orthogonally diagonalisable
    - Consider n=k+1
      * Let be an eigenvalue and the corresponding eigenvector
      * Since A is real, those two are real too
      * Assume is a unit vector (because it can always be normalised)
      * Using Gram-Schmidt, extend to an orthonormal basis
      * Let be the orthogonal matrix with column vectors found previously
        + B is symmetric because of the diagonal consisting of unit vector scalar products (1) and off-diagonal entries consisting of orthogonal vector scalar products (0)
      * Then is symmetric
      * Then there is a matrix s. t.
      * Then let be the block matrix [P2 in lower right corner, 1 in upper left, 0’s everywhere else)
      * Since is orthogonal, so too is
      * Then is diagonal
    - Thus, by induction, true

Quadratic Form:

* A quadratic form in n variables is a function of the form
* where A is a symmetric n x n matrix and . A is **the matrix associated with f**.
* Remarks:
  + Every quadratic form in 1 variable has the form
  + An example of a quadratic form in 2 variables is
  + A is not unique – two different A’s can lead to the same function f: is not injective
  + Every A defined a quadratic form f: is surjective

Equivalence:

* Two quadratic forms in n variables are equivalent if there exists an n x n matric P such that for all

Principal Axes Theorem:

* Every quadratic form can be diagonalised. If A is an n x n symmetric matric such that there exists a quadratic form , and if Q is an orthogonal matrix, then the change of variables transforms the quadratic form q into which has no cross product terms. If the eigenvalues of A are and , then
* Proof:
  + Let A be a symmetric n x n matrix associated with q
  + Then there exists an orthogonal n x n matrix Q such that
  + Let or
  + (by substitutions), which is a quadratic form with no cross terms (since D is diagonal)

Rank, Signature:

* Let Q be a non-zero quadratic form in n variables that transforms such that , A is a symmetric n x n matric with positive eigenvalues and negative eigenvalues. Then
  + is the **rank** of q
  + is the **signature** of q

# Week 10

## 10b: Quadratic Forms Continued

A quadratic form in n variables is classified as **one** of the following:

1. *positive definite* if for all
2. *positive semidefinite* if for all x
3. *negative definite* if for all
4. *negative semidefinite* iffor all x
5. *indefinite* if f(x) takes on both positive and negative values

If A is an n x n symmetric matrix, the quadratic form is

1. positive definite if and only if all the eigenvalues of A are positive (signature is n)
2. positive semidefinite if and only if all the eigenvalues of A are non-negative (signature = rank)
3. negative definite if and only if all the eigenvalues of A are negative (signature is -n)
4. negative semidefinite if and only if all the eigenvalues of A are non-positive (signature=-rank)
5. indefinite if and only if A has both positive and negative eigenvalues (- rank < signature < rank)

Conic sections:

* represents a conic section for ,
  + **Ellipse** if both eigenvalues are **positive**
  + **No graph** if both eigenvalues are **negative**
  + **Hyperbola** if eigenvalues have **different** **signs**

## 10c: Hermitian Matrices

Let A be a complex n x n matrix with We define its complex conjugate by . The **conjugate transpose** of A is the matrix

* Properties (if are matrices and ):
  + - Proof:
      * LHS:
      * Write and . Let be the (ij)-th entry of the product AB:
      * Hence, the (ij)th entry of is
      * RHS:
      * Let be the (ij)-th entry of the product
      * …

Let and be vectors in . Then the **complex dot product** of u and v is defined by

Property:

A square complex matrix A is called **Hermitian** (**self-adjoint**) if

* Visible properties:
  + Real entries in diagonal
  + Symmetric entries are complex conjugates
* Properties:
  + Every eigenvalue of a Hermitian matrix is real
  + The eigenvectors corresponding to distinct eigenvalues of A are orthogonal

## 10d: Unitary Matrices

A square complex **matrix** U is called **unitary** if .

* The following are equivalent:
  + U is a unitary matrix
  + The columns of U form an orthonormal basis for with respect to the complex dot product
  + for every
  + for every

A square complex matrix A is **unitarily diagonalisable** if there exists a unitary matrix U and diagonal matrix D such that

Properties:

* Every Hermitian matrix is unitarily diagonalisable
* A square complex matrix A is unitarily diagonalisable if and only if

Definitions:

* A square complex matrix A is called **normal** if
* A skew Hermitian matrix:

Theorem: Every Hermitian matrix, every unitary matrix, and every skew Hermitian matrix is normal

* Analogy with reals: refers to symmetric, orthogonal and skew-symmetric matrices, respectively
* Proof:
  + Hermitian:
    - Let A be a square Hermitian matrix, so
    - Then
  + Unitary:
    - Let A be a unitary matrix, so
    - Then , so
  + Skew Hermitian:
    - Let A be a skew Hermitian matrix, so
    - Then